REGIONALLY ANALYTIC METHOD IN PROBLEMS OF OPTIMUM CONTROL OF NONSTATIONARY THERMAL MODES

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Regionally-analytic solutions and control functions are obtained for problems of most rapid heating of bodies under given constraints.

A survey of the literature shows that there are still problems whose solution would permit significant progress along the path of raising the efficiency of applied optimal fast-response control of the temperature modes of solids [1-3]. This also refers to appropriate one-dimensional nonstationary problems for simple bodies [3].

One of these problems is the development of such analytic approaches as would allow regionally-analytic solutions in time to be obtained for appropriate problems and control functions in the form of compact formulas based on elementary functions.

This paper is indeed devoted to an investigation of these questions where a regionally-analytic method is proposed for solving optimal fast-response control problems of the heating and cooling of solids under given constraints on the control, the temperature in separate parts of the body, the heating rate, and the temperature field gradients.

Regionally-analytic structures with respect to time that satisfy a given boundary condition of the third kind for any time are constructed in this paper. The control function enters one of the components of the solution structure explicitly. The permits a differential equation for the control function to be obtained for each of the time regions under appropriate conditions and also simple transcendental equations to determine the control switching time, to determine the disconnection times and the regionally-analytic control functions with respect to the time.

1. Construction of regional body heating control with respect to time under constraints on the control and distribution functions. Let us consider the control problem for heating a plate in the following formulation

$$\frac{\partial T(x,\ \mathrm{Fo})}{\partial\,\mathrm{Fo}} = \frac{\partial^2 T(x,\ \mathrm{Fo})}{\partial x^2}, \tag{1}$$

$$\left(\pm\frac{\partial T(x,\ \mathrm{Fo})}{\partial x} + \mathrm{Bi}(\mathrm{Fo})\,T(x,\ \mathrm{Fo})\right)\Bigg|_{x=\pm l} = \mathrm{Bi}(\mathrm{Fo})\,t_{\mathrm{av}}(\mathrm{Fo}), \tag{2}$$

$$T(x,\ 0) = \psi(x), \tag{3}$$

$$T(\pm l,\ \mathrm{Fo}_*) = \beta t_0, \tag{4}$$

$$T(\pm l,\ \mathrm{Fo}) \leqslant f(\mathrm{Fo})\,t_0, \tag{5}$$

$$T(\pm l,\ \mathrm{Fo}_*) - T(0,\ \mathrm{Fo}_*) \leqslant \gamma t_0, \tag{6}$$

where $\psi(x)$, $f(\mathrm{Fo})$, $\mathrm{Bi}(\mathrm{Fo})$ are given functions, β, γ are given coefficients in the conditions (3)-(6), and t_0 is the upper limit of the allowable control.

In this case a control function

$$f(\mathrm{Fo})\,t_0 < t_{\mathrm{av}}(\mathrm{Fo}) \leqslant t_0,$$

must be found as as to heat it in the minimally short time Fo_* under constraints on the maximal body temperature (4) by using the mechanism of convective heat transfer from the initial (3) to the final state $T(x,\ \mathrm{Fo}_*)$ with conditions (4) and (6) taken into account. The

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constraint (5) is here imposed on the temperature mode during heating.

This problem is solved in [3] for $f(\mathrm{Fo}) = 0.5 + 0.2\mathrm{Fo}$; $\psi(x) = 0$; $\mathrm{Bi} = 1.5$; $\beta = 0.8$; $\ell = 1$; $\gamma = 0.05$ by using the analytic method, traditional in mathematical physics of solving the heat conduction equation. Because of the awkwardness of the analytic solutions obtained, determination of the control switching times and the disconnection times as well as construction of the multistage control are realized by using an electronic computer.

For the values of $f(\mathrm{Fo})$, $\psi(x)$, Bi, β, γ given above the analytic solution of the problem (1)-(3) in a first approximation that satisfies condition (2) exactly for any time for the maximal value of the temperature $t_m(\mathrm{Fo}) = 1$ is constructed in the form

$$T_1(x,\ \mathrm{Fo}) = 1 + C_0^{(1)}(\mathrm{Fo})\,[1 + 0{,}75\,(1 - x^2)]. \tag{7}$$

Substituting the function (7) into (1) and using the method of orthogonal projections [4], we obtain for the function $C_0^{(1)}(\mathrm{Fo})$: $\dfrac{dC_0^{(1)}(\mathrm{Fo})}{d\,\mathrm{Fo}} + C_0^{(1)}(\mathrm{Fo}) = 0$; $C_0^{(1)}(\mathrm{Fo}) = d_1 \exp(-\mathrm{Fo})$.

We find $d_1 = -2/3$ from the condition

$$\int_0^1 T_1(x,\ 0)\,dx = 0.$$

The condition

$$T_1(\pm 1,\ \mathrm{Fo}_1) = 0{,}5 + 0{,}2\,\mathrm{Fo}_1. \tag{8}$$

is used to determine the control switching time Fo_1. Solving this equation, we obtain $\mathrm{Fo}_1 = 0.51$, where $\mathrm{Fo}_1 = 0.466$ according to the data in [3].

We construct the function $T_2(x,\ \mathrm{Fo})$ for the second time region that satisfies condition (2) exactly for the unknown function $t_{2m}(\mathrm{Fo})$ in the form

$$T_2(x,\ \mathrm{Fo}) = t_{2\mathrm{av}}(\mathrm{Fo}) + C_0^{(2)}(\mathrm{Fo})\,[1 + 0.75\,(1 - x^2)]. \tag{9}$$

Taking into account the condition $T_2(\pm 1,\ \mathrm{Fo}) = 0.5 + 0.2\mathrm{Fo}$, we determine the function $t_{2m}(\mathrm{Fo})$ in the form $t_{2m}(\mathrm{Fo}) = 0.5 + 0.2\mathrm{Fo} - C_0^{(2)}(\mathrm{Fo})$.

The appropriate differential equation for the function $C_0^{(2)}(\mathrm{Fo})$ is derived analogously. Substituting the exact solution of this equation into (9), we obtain

$$T_2(x,\ \mathrm{Fo}) = 0.5 + 0.2\,\mathrm{Fo} + 0.75\,(1 - x^2)\,[d_2 \exp(-3\,\mathrm{Fo}) - 0.133]. \tag{10}$$

The coefficient d_2 is determined from the condition

$$\int_0^1 [T_{k+1}(x,\ \mathrm{Fo}_k) - T_k(x,\ \mathrm{Fo}_k)]\,dx = 0 \tag{11}$$

for $k = 1$, we hence obtain $d_2 = -1.2497$. Then the function $t_{2m}(\mathrm{Fo})$ is converted to the form

$$t_{2\mathrm{av}}(\mathrm{Fo}) = 0.633 + 0.2\,\mathrm{Fo} + 1.2497 \exp(-3\,\mathrm{Fo}). \tag{12}$$

The control switching time Fo_2 is determined from the condition

$$T_2(1,\ \mathrm{Fo}_2) = 0{,}8 = 0{,}5 + 0{,}2\,\mathrm{Fo}_2.$$

In this case as in [3], $\mathrm{Fo}_2 = 1.5$.

We construct the function $T_3(x,\ \mathrm{Fo})$ for the third time region in the form (9)

$$T_3(x,\ \mathrm{Fo}) = t_{3\mathrm{av}}(\mathrm{Fo}) + C_0^{(3)}(\mathrm{Fo})\,[1 + 0.75\,(1 - x^2)]. \tag{13}$$

The constraint (4) yields for the function $t_{3m}(\mathrm{Fo})$

$$t_{3\mathrm{av}}(\mathrm{Fo}) = 0.8 - C_0^{(3)}(\mathrm{Fo}). \tag{14}$$

Applying the method of orthogonal projections to (1) for the function (13), solving the appropriate differential equations for the function $C_0^{(3)}(\mathrm{Fo})$ exactly and substituting the result into (13), we obtain

$$T_3(x,\ \mathrm{Fo}) = 0{,}8 + 0{,}75 d_3 \exp(-3\,\mathrm{Fo})\,(1 - x^2). \tag{15}$$

We find the undetermined coefficient d_3 from the condition (11) for $k = 2$. Substituting the result into (14) and (15) we obtain

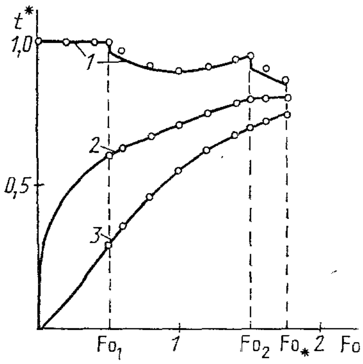

Fig. 1

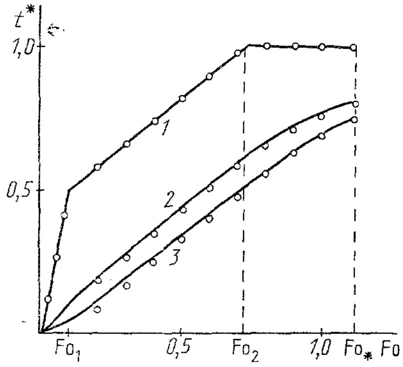

Fig. 2

Fig. 1. Three-stage control $t^* = t_m(\text{Fo})t_0^{-1}$ (curve 1) for plate heating and the plate temperature $T(1;\ \text{Fo})t_0^{-1}$, $T(0;\ \text{Fo})t_0^{-1}$ (curves 2 and 3).

Fig. 2. Three-stage control $t^* = t_{av}(\text{Fo})t_0^{-1}$ (curve 1) for plate heating and plate temperature $T(0,\ 5;\ \text{Fo})t_0^{-1}$, $T(0;\ \text{Fo})t_0^{-1}$ (curves 2 and 3).

$$t_{3av}(\text{Fo}) = 0.8 + 13.2489 \exp(-3\,\text{Fo}), \tag{16}$$

$$T_3(x,\ \text{Fo}) = 0.8 - 9{,}9367 \exp(-3\,\text{Fo})\,(1 - x^2). \tag{17}$$

Taking condition (6) into account we arrive at the equation

$$-3\,\text{Fo}_* = \ln\left(\frac{5}{993.67}\right) \tag{18}$$

to determine the control disconnection time, where $\text{Fo}_* = 1.764$ from the solution of (8), and $\text{Fo}_* = 1.794$ according to the data in [3].

Curve 1 in Fig. 1 characterizes the three-stage control of plate heating. The plate temperature $T(1,\ \text{Fo})t_0^{-1}$ and $T(0,\ \text{Fl})t_0^{-1}$ is shown by curves 2 and 3. The results obtained from (7), (10), (12), (16), and (17) are represented by points in Fig. 1.

2. Construction of the body heating control, regional in time, under constraints on the control and the temperature field gradient. In this case, for given $\text{Bi} = \text{Bi}(\text{Fo})$, $\psi(x)$, the condition

$$\left.\frac{\partial T}{\partial x}\right|_{x=\pm l} = \mp \eta t_0 \tag{19}$$

and the condition on the upper bound of the allowable control $t_m(\text{Fo}) = u_2(\text{Fo})$

$$u_2(\text{Fo}) = \begin{cases} 5\,\text{Fo}, & \text{Fo} \in [0;\ 0.2], \\ 1, & \text{Fo} \in (0{,}2;\ \text{Fo}_*) \end{cases} \tag{20}$$

with a constraint on the temperature drop are appended to equations (1)-(4).

The problem is solved in [3] for $t = 0.5$; $\text{Bi} = 1$; $\beta = 0.8$; $\eta = 0.4$; $\psi(x) = 0$ by the same method as in the preceding problem.

For the first time region the function $T_1(x,\ \text{Fo})$ satisfying condition (2) exactly with the condition (20) $t_{1m}(\text{Fo}) = 0.5\text{Fo}$ taken into account is constructed in the form

$$T_1(x,\ \text{Fo}) = 5\,\text{Fo} + C_0^{(1)}(\text{Fo})\,[1{,}25 - x^2]. \tag{21}$$

Substituting $T_1(x,\ \text{Fo})$ into (1) and applying the method of orthogonal projections, we obtain for the function $C_0^{(1)}(\text{Fo})$

$$\frac{dC_0^{(1)}(\text{Fo})}{d\,\text{Fo}} + \frac{6}{7}C_0^{(1)}(\text{Fo}) + \frac{30}{7} = 0. \tag{22}$$

The determination of $C_0^{(1)}(\text{Fo})$ from (22) is performed with condition (3) taken into account. Then

$$T_1(x,\ \mathrm{Fo}) = 5\,\mathrm{Fo} + 5\left[\exp\left(-\frac{6}{7}\mathrm{Fo}\right) - 1\right](1.25 - x^2). \tag{23}$$

The control switching time Fo_1 is determined from the solution of the equation

$$1 - \exp\left(-\frac{6}{7}\mathrm{Fo}_1\right) = 0{,}08.$$

by using condition (19). In this case $\mathrm{Fo}_1 = 0.097$ whereas $\mathrm{Fo}_1 = 0.1$ according to the data of [3].

For the second time region the function $t_{2m}(\mathrm{Fo})$ is desired, consequently taking (21) into account

$$T_2(x,\ \mathrm{Fo}) = t_{2\mathrm{av}}(\mathrm{Fo}) + C_0^{(2)}(\mathrm{Fo})(1.25 - x^2). \tag{24}$$

According to condition (19)

$$\left.\frac{\partial T_2}{\partial x}\right|_{x=\pm 0,5} = -C_0^{(2)}(\mathrm{Fo}) = 0.4.$$

Then, substituting the function (24) into (1) and applying the method of orthogonal projections, we obtain for the function $t_{2m}(\mathrm{Fo})$

$$\frac{dt_{2\mathrm{av}}(\mathrm{Fo})}{d\,\mathrm{Fo}} = 0{,}8;\ t_{2\mathrm{av}}(\mathrm{Fo}) = 0.8\,\mathrm{Fo} + \beta_2. \tag{25}$$

From condition (11) for $k = 1$; $0 \le x \le 0.5$ $\beta_2 = 0.4087$.

The control switching time Fo_2 is determined with condition (20) taken into account

$$t_{2\mathrm{av}}(\mathrm{Fo}_2) = 0.8\,\mathrm{Fo}_2 + 0{,}4087 = 1. \tag{26}$$

According to the data of [3] $\mathrm{Fo}_2 = 0.72$ while $\mathrm{Fo}_2 = 0.737$ from (26).

Taking account of condition (20) we construct the function $T_3(x,\ \mathrm{Fo})$ for the third time region for $t_{3m}(\mathrm{Fo}) = 1$ in the form

$$T_3(x,\ \mathrm{Fo}) = 1 + C_0^{(3)}(\mathrm{Fo})(1.25 - x^2). \tag{27}$$

As in the previous cases, applying the method of orthogonal projections to (1) and (27) analogously, and substituting the result for the function $C_0^{(3)}(\mathrm{Fo})$ into (27) we find

$$T_3(x,\ \mathrm{Fo}) = 1 + \beta_3 \exp\left(-\frac{12}{7}\mathrm{Fo}\right)(1.25 - x^2). \tag{28}$$

For $k = 2$, $0 \le x \le 0.5$, from condition (1)

$$\beta_3 = -0{,}4\exp\left(\frac{12}{7}\mathrm{Fo}\right).$$

Applying condition (4) for $\ell = 0.5$, we obtain the equation

$$\exp\left(\frac{12}{7}\mathrm{Fo}_2 - \mathrm{Fo}_*\right) = 0{,}5 \tag{29}$$

to determine the control switching time. Solving (29), we find $\mathrm{Fo}_* = 1.14$ while $\mathrm{Fo}_* = 1.11$ from the data in [3].

Curve 1 in Fig. 2 characterizes the three-stage control of plate heating. The plate temperature $T(0.5;\ \mathrm{Fo})t_0^{-1}$ and $T(0;\ \mathrm{Fo})t_0^{-1}$ is shown by curves 2 and 3. Data obtained from (23)-(25), (28) are represented by points.

3. Construction of a body heating control, regional in time, under constraints on the control and the heating rate. In this case, (1)-(3) are included in the problem under consideration under constraints on the control

$$0 \leqslant t_{\mathrm{av}}(\mathrm{Fo}) \leqslant u_2(\mathrm{Fo})$$

and the heating rate

$$\frac{\partial T(l,\ \mathrm{Fo})}{\partial\,\mathrm{Fo}} \leqslant \mu t_0 \tag{30}$$

with the final heating goal

$$\max_{x\in[0,1]} T(x,\ \mathrm{Fo}_*) = \beta t_0, \tag{31}$$

$$\max_x T(x,\ \mathrm{Fo}_*) - \min_x T(x,\ \mathrm{Fo}_*) \leqslant \gamma t_0;\ x \in [0,\ 1]. \tag{32}$$

This problem is solved in [3] for $\ell = 1$; $\mathrm{Bi} = 0.5$; $\beta = 0.8$; $\mu = 0.175$; $\gamma = 0.02$; $\psi(x) = 0$,

$$u_2(\mathrm{Fo})\,t_0^{-1} = \begin{cases} 0{,}5\,\mathrm{Fo}, & 0 \leqslant \mathrm{Fo} \leqslant 2, \\ 1, & \mathrm{Fo} > 2 \end{cases} \tag{33}$$

by the same method as was presented above.

We construct the function $T_1(x,\ \mathrm{Fo})$ for the first time region in a first approximation that satisfies condition (2) exactly for $t_m(\mathrm{Fo}) = 0.5\mathrm{Fo}$ (condition (33)) in the form

$$T_1(x,\ \mathrm{Fo}) = 0{,}5\,\mathrm{Fo} + C_0^{(1)}(\mathrm{Fo})\,(1{,}25 - 0{,}25x^2). \tag{34}$$

We find the function $C_0^{(1)}(\mathrm{Fo})$ from the solution of the appropriate differential equation obtained by applying the method of orthogonal projections to (1) for the function (34). Then (34) is converted as follows

$$T_1(x,\ \mathrm{Fo}) = 0{,}5\,\mathrm{Fo} + 0.25\left[\beta_1 \exp\left(-\frac{3}{7}\mathrm{Fo}\right) - 1\right](5 - x^2). \tag{35}$$

It is easy to verify that the initial condition (3) is satisfied exactly for $\beta_1 = 1$.

Applying the condition (30), we find for the control switching time Ro_1

$$0{,}325 = \frac{7}{3}\ \exp\left(-\frac{3}{7}\,\mathrm{Fo}_1\right);\ \mathrm{Fo}_1 = 0.645.$$

where $\mathrm{Fo}_1 = 0.65$ according to the data in [3].

For the second time region the solution of the problem in a first approximation for unknown functions $t_{2m}(\mathrm{Fo})$ and $C_0^{(2)}(\mathrm{Fo})$ satisfying condition (2) exactly, is constructed in the form

$$T_2(x,\ \mathrm{Fo}) = t_{2\mathrm{av}}(\mathrm{Fo}) + 0{,}25C_0^{(2)}(\mathrm{Fo})\,(5 - x^2). \tag{36}$$

Using condition (30), we obtain the equation

$$\frac{dt_{2\mathrm{av}}(\mathrm{Fo})}{d\,\mathrm{Fo}} = -0{,}175 - \frac{dC_0^{(2)}(\mathrm{Fo})}{d\,\mathrm{Fo}}. \tag{37}$$

We find the function $C_0^{(2)}(\mathrm{Fo})$ from the solution of the corresponding differential equation obtained by application of the method of orthogonal projections to (1) for the function (36). Formulas (36) and (37) are here converted as follows

$$T_2(x,\ \mathrm{Fo}) = t_{2\mathrm{av}}(\mathrm{Fo}) + 0.25\,[\beta_2 \exp(-3\,\mathrm{Fo}) - 0{,}35]\,(5 - x^2), \tag{38}$$

$$\frac{dt_{2\mathrm{av}}(\mathrm{Fo})}{d\,\mathrm{Fo}} = 0.175 + 3\beta_2 \exp(-3\ \mathrm{Fo}). \tag{39}$$

Applying the condition $t_{1m}(\mathrm{Fo}_1) = t_{2m}(\mathrm{Fo}_1)$ we write for the function $t_{2m}(\mathrm{Fo})$

$$t_{2\mathrm{av}}(\mathrm{Fo}) = 0.175\,\mathrm{Fo} + \beta_2\,[0.144 - \exp(-3\,\mathrm{Fo})] + 0.21. \tag{40}$$

For $k = 1$ from condition (11) there follows that $\beta_2 = 0.75$. The control switching time Fo_2 is determined from the solution of the equation $t_{2m}(\mathrm{Fo}_2) = 1$ by using condition (33), hence, we obtain $\mathrm{Fo}_2 = 3.91$, whereas $\mathrm{Fo}_2 = 3.86$ according to the data of [3].

For the third time region $t_{3m}(\mathrm{Fo}) = 1$ from condition (33), then taking account of (34)

$$T_3(x,\ \mathrm{Fo}) = 1 + 0.25C_0^{(3)}(\mathrm{Fo})\,(5 - x^2). \tag{41}$$

The function $C_0^{(3)}(\mathrm{Fo})$ is found analogously to the function $C_0^{(2)}(\mathrm{Fo})$. In this case the undetermined coefficient β_3 is determined from condition (11) for $k = 2$, then (41) is converted to the form

$$T_3(x,\ \mathrm{Fo}) = 1 - 0.4655 \exp\left(-\frac{3}{7}\,\mathrm{Fo}\right)(5 - x^2). \tag{42}$$

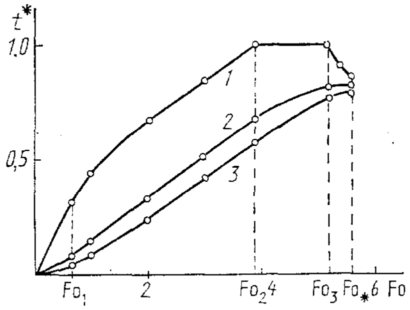

Fig. 3. Four-stage plate heating control $t^* = t_m(\mathrm{Fo})t_0^{-1}$ (curve 1) and plate temperature $T(1;\ \mathrm{Fo})t_0^{-1}$ and $T(0;\ \mathrm{Fo})t_0^{-1}$ (curves 2 and 3).

Taking account of condition (31), we obtain $T_3(1;\ \mathrm{Fo}_3) = 0.8$; $\mathrm{Fo}_3 = 5.206$ from condition (31), while $\mathrm{Fo}_3 = 5.2$ from the data in [3].

For the fourth time region we represent the approximate solution of the problem for the unknown function $t_{4m}(\mathrm{Fo})$ in the form

$$T_4(x,\ \mathrm{Fo}) = t_{4\mathrm{av}}(\mathrm{Fo}) + 0{,}25C_0^{(4)}(\mathrm{Fo})(5 - x^2). \tag{43}$$

Condition (31) permits the dependence

$$t_{4}\mathrm{av}(\mathrm{Fo}) = 0.8 - C_0^{(4)}(\mathrm{Fo}),$$

to be obtained, then

$$\frac{dt_{4}\mathrm{av}(\mathrm{Fo})}{d\,\mathrm{Fo}} = -\frac{dC_0^{(4)}(\mathrm{Fo})}{d\,\mathrm{Fo}}. \tag{44}$$

Applying condition (44) and the method of orthogonal projections to (1) for the function (43), we obtain for the functions $C_0^{(4)}$, t_{4m}, T_4

$$\frac{dC_0^{(4)}(\mathrm{Fo})}{d\,\mathrm{Fo}} + 3C_0^{(4)}(\mathrm{Fo}) = 0,$$

$$t_{4}\mathrm{av}(\mathrm{Fo}) = 0.8 - \beta_4 \exp(-3\,\mathrm{Fo}), \tag{45}$$

$$T_4(x,\ \mathrm{Fo}) = t_{4\mathrm{av}}(\mathrm{Fo}) + 0.25\beta_4 \exp(-3\,\mathrm{Fo})(5 - x^2). \tag{46}$$

From the condition of equality of the temperature at the time Fo_3 we obtain

$$\beta_4 = -0.2 \exp(3\,\mathrm{Fo}_3).$$

We apply condition (32) to determine the control disconnection time Fo_*, then

$$3(\mathrm{Fo}_3 - \mathrm{Fo}_*) = \ln 0.4;\ \ \mathrm{Fo}_* = 5.51.$$

while $\mathrm{Fo}_* = 5.6$ according to the data in [3].

Curve 1 in Fig. 3 characterizes four-stage plate heating control. The plate temperature $T(1;\ \mathrm{Fo})/t_0$ and $T(0;\ \mathrm{Fo})/t_0$ is shown by curves 2 and 3. The results obtained from (35), (38), (40), (42), (45), (46) are represented by points.

The proposed method can also efficiently construct optimal multistage fast-response control of the nonstationary thermal mode of continuous and hollow cylinders and spheres under the constraints considered above.

LITERATURE CITED

1. R. Bellman, I. Klinsberg, and O. Gross. Certain Questions of the Mathematical Theory of Control Processes [Russian translation], Moscow (1962).
2. A. G. Butkovskii, Control Methods for Systems with Distributed Parameters [in Russian], Moscow (1975).
3. V. M. Vigak, Optimal Control of Nonstationary Temperature Modes [in Russian], Kiev (1979).
4. B. P. Demidovich, I. A. Maron, and E. Z. Shuvalova, Numerical Analysis Methods [in Russian], Moscow (1963).